

Closed Trapped Surfaces in Cosmology

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Abstract

The existence of closed trapped surfaces need not imply a cosmological singularity when the spatial hypersurfaces are compact. This is illustrated by a variety of examples, in particular de Sitter spacetime admits many closed trapped surfaces and obeys the null convergence condition but is non-singular in the $k=+1$ frame.

1 Introduction

Since Roger Penrose' pioneering paper of 1965 [1], the existence of closed trapped surfaces ('CTSs') has been understood as a geometrical condition that, jointly with suitable energy conditions, in many circumstances leads to the existence of space-time singularities in the context of both gravitational collapse and cosmology. This understanding has been codified in the series of singularity theorems proved by Penrose and Stephen Hawking [1, 2, 3]. The various theorems involve different combinations of geometric requirements and energy conditions. Under the assumptions of standard hot big bang theory, these conditions will indeed be met in the cosmological context, because the existence of the black-body cosmic background radiation implies the existence of CTSs in the era between decoupling and the present day [4, 2, 3] and so leads to prediction of a (classical) singularity at the start of the universe. However it is now known that scalar fields can violate some of the energy conditions, thereby providing the foundation of the inflationary universe paradigm [5]. The possibility then arises of singularity avoidance in realistic early universe models because of these energy condition violations, and indeed even avoidance of a quantum gravity regime is possible [10], despite the existence of CTSs. The purpose of this paper is to revisit the relation between CTSs and spacetime singularities in this cosmological context, characterising cases where existence of CTSs do not imply the existence of singularities. In particular we examine the case of the de Sitter Universe, showing that CTSs exist in these spacetimes even though (in maximally extended form) they are both geodesically complete and stable to perturbations.

1.1 Closed Trapped Surfaces

Consider a spacelike 2-surface S with spherical topology. It is a *past closed trapped surface* if both families of past null geodesics orthogonal to the 2-surface S are converging, i.e. if their divergences (evaluated at the surface) are negative. Similarly *future closed trapped surfaces* occur if we replace "past" by "future" in the above. Both past and future closed trapped surfaces will be referred to as *closed trapped surfaces* ('CTSs'). We will in this paper be concerned with 2-surfaces that are 2-spheres with a group of isometries transitive on them, so they are homogeneous 2-dimensional subspaces of spacetime. Then the value of the divergence of each family of normals is constant over the 2-surface, and can be characterised by a single number on each 2-sphere. Thus we will in fact be considering existence of homogeneous closed trapped 2-spheres.

Marginally closed trapped surfaces exist if the divergences are non-positive for these families of null geodesics; that is, if the divergences are either zero or negative rather than strictly negative.

1.2 Energy Conditions

Energy conditions generically lead to convergences of irrotational families of non-spacelike and null geodesics respectively.

1.2.1 Non-spacelike convergence condition

This is the condition

$$R_{ab}K^aK^b \geq 0 \text{ for all non-spacelike vectors } K^a.$$

For perfect fluids, this translates into $\mu + p \geq 0$, $\mu + 3p \geq 0$, which will be true for all ordinary matter. For scalar fields, it becomes $\frac{1}{2}\dot{\phi}^2 \geq 0$, $\dot{\phi}^2 \geq V(\phi)$, hence is violated when the slow rolling condition $\dot{\phi}^2 \ll V(\phi)$ is satisfied. A cosmological constant is the case $\dot{\phi}^2 = 0$, $V > 0$ and hence violates this condition.

1.2.2 Null convergence condition

This is the condition

$$R_{ab}K^aK^b \geq 0 \text{ for all null vectors } K^a \tag{1}$$

which is implied by the previous:

$$\{\text{Non-spacelike convergence condition}\} \Rightarrow \{\text{Null convergence condition}\}.$$

For perfect fluids, this translates into $\mu + p \geq 0$, which will be true for all ordinary matter. For scalar fields, it is $\frac{1}{2}\dot{\phi}^2 \geq 0$, and so is true for all ordinary scalar fields (we discount the possibility of 'phantom matter' that violates this condition, see Gibbons [7] for a discussion).

1.3 Focussing

The equation determining the evolution of the convergence $\theta = K^a_{;a}$ of hypersurface-orthogonal null geodesics is

$$\frac{d\theta}{dv} + \frac{1}{2}\theta^2 = -R_{ab}K^aK^b - 2\sigma^2, \quad (2)$$

while the shear propagation equation is

$$\frac{d}{dv}\sigma_{mn} = -\theta\sigma_{mn} - C_{manb}K^aK^b.$$

This shows that the shear can only remain zero either for very special spacetimes (e.g. Robertson-Walker spacetimes where $C_{manb} = 0$), or for very special null rays in a more generic spacetime, so that $C_{manb}K^aK^b = 0$ at every point on the null geodesics because the geodesic tangent vector is in a special relation to the Weyl tensor (it is a principal null direction). Once the shear is non-zero, it acts as a source term in the null Raychaudhuri equation (2).

When the null convergence condition (1) is true, there is an exceptional case and a generic case for families of hypersurface orthogonal null geodesics. The exceptional case occurs if $\theta = 0$, i.e. no focussing occurs:

$$\theta = 0 \Rightarrow R_{ab}K^aK^b = 0, \quad \sigma^2 = 0 \Rightarrow C_{acbd}K^aK^b = 0.$$

This cannot be true in a generic cosmological context, for example a perturbed Robertson-Walker universe, when both the Ricci and Weyl tensor conditions will be violated along a generic null ray. The generic case is when either $\theta_0 < 0$ or $\theta_0 > 0$. Both imply $\theta \rightarrow -\infty$ within a finite affine distance, either to the past or the future. Then the null rays intersect at a caustic, so the surface generated by the null geodesics experiences self-intersections either before or at those events.

Hence when we consider CTSs in realistic cosmologies, which will always satisfy the null convergence condition, both families of converging orthogonal null geodesics, which generate the boundary of their past for the case of past CTSs, will self-intersect. Then by well-known causal theorems, these null rays will lie from then on inside the pasts of the CTSs; the boundary of the past is therefore compact. This is what underlies the singularity theorems.

1.4 The Major Singularity Theorems

The major singularity theorems referring to CTSs are given in Hawking and Ellis [3] ('HE'). Each case assumes a CTS or roughly equivalent condition, plus the following:

HE Theorem 1 [1] - A non-compact Cauchy surface and the null convergence condition,

HE Theorem 2 [2] - The non-spacelike convergence condition and a causality condition,

HE Theorem 3 [8] - The non-spacelike convergence condition and a causality condition.

Roughly speaking: in each case the boundary of the past comes to an end because of existence of self-intersections points in its generating geodesics; but the past is contained within the boundary, hence a singularity must occur.

We now consider how these theorems apply to Friedmann-Lemaitre ('FL') model universes, interpreted here as universes with a Robertson-Walker geometry and matter content of ordinary matter and/or a scalar field.

2 Friedmann-Lemaitre Models

Friedmann-Lemaitre universe have a Robertson-Walker ('RW') metric which can be represented in the form

$$ds^2 = -dt^2 + S^2(t) (dr^2 + f^2(r)(d\theta^2 + \sin^2 \theta d\phi^2)) \quad (3)$$

The scale factor is $S(t)$, the matter 4-velocity is: $u^a = \delta_0^a \Rightarrow u_a = g_{ab}u^b = -\delta_a^0$, and $f(r) = (\sin r, r, \sinh r)$ if $(k = 1, 0, -1)$ respectively.

The metric determinant g is

$$g = -S^6(t)f^4(r)\sin^2 \theta \Rightarrow \sqrt{-g} = S^3(t)f^2(r)\sin \theta.$$

Note that every 2-surface $\mathcal{S}(t, r) : (t = \text{const}, r = \text{const})$ is a homogeneous 2-sphere of area

$$A = 4\pi S^2(t)f^2(r).$$

2.1 Radial Null Geodesics

The family of past-directed radial null geodesics in this space-time has tangent vector field

$$K^a = \frac{1}{S(t)} \left(-1, \pm \frac{1}{S(t)}, 0, 0 \right) = \frac{dx^a}{dv}$$

[9] where the sign depends on whether the geodesics are ingoing or outgoing. This form gives

$$K^a K_a = g_{ab}K^a K^b = 0,$$

$$K^a u_a = \frac{1}{S(t)} = (1 + z)$$

as required, and they are normal to the family of instantaneous homogeneous 2-spheres $\mathcal{S}(t, r)$ because they lie in the orthogonal 2-plane ($\theta = \text{const}, \phi = \text{const}$) to these 2-spheres, described by coordinates (t, r) . They diverge from the point of origin of coordinates $r = 0$, and refocus at the antipodal point $r = \pi$ when $k = +1$.

The divergence of this family of null geodesics is given by

$$K_{;a}^a = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} (\sqrt{-g} K^a) = \frac{2}{S^2(t)} \left[-\dot{S}(t) \pm \frac{\partial f(r)/\partial r}{f(r)} \right]. \quad (4)$$

CTSs occur if the divergence is negative for both families of null geodesics for some value of r and t , i.e. for both choices of sign in the last term on this 2-surface. We now look at a series of specific cases.

3 The Early and Late Universe

3.1 Standard radiation/matter dominated expansion

Here $S(t) = at^n$ with $a > 0$ and $n = 2/3$ in the matter era, $n = 1/2$ in the radiation era, and with $0 < t < \infty$. The divergence of the radial families of null geodesics is given by (4):

$$K_{;a}^a = \frac{2}{at^{2n}} \left[-ant^{n-1} \pm \left(\cot r, \frac{1}{r}, \coth r \right) \right]$$

for $k = +1, 0, -1$ respectively. These expressions give the divergences of the normals of the 2-spheres (r, t) constant. For each $t > 0$ and each value of k these will be CTSs, obtained by choosing r large enough that the magnitude of the second term in the square brackets is less than that of the first term in these brackets, so that $K_{;a}^a < 0$ for both signs (i.e. for both ingoing and outgoing null geodesics). Note that previous examinations of these surfaces have concentrated on showing that the past light cone of any observer will start reconverging and thus that closed trapped surfaces will occur in the past of the observer, associated with this refocussing of their past light cone. What is shown here is a bit different, namely that for any value of t , if one goes to large enough values of r there will be closed trapped surfaces surrounding the origin at that time. Hence there are closed trapped surfaces surrounding us even today (associated with the reconvergence of the past light cone of observers in our future).

Here the energy condition $\mu + 3p > 0$ is satisfied because we consider only ordinary matter. If this energy condition does not change in the past then a singularity is predicted in all cases via the Raychaudhuri equation for exact FL models [10], and via Theorems 2&3 above for universes that are perturbed FRW models at late times even if they are quite different at early times (small enough perturbations will necessarily preserve the inequalities $K_{;a}^a < 0$ in the late-time era), and also by Theorem 1 in the case $k \neq 1$. Hence the existence of singularities is predicted and is stable to perturbations of these models at late times (which may correspond to very large changes in the models at early enough times) [3].

3.2 Inflation

In realistic universes the Hot Big Bang era may be preceded by an era of inflation. It will still be true that CTSs occur in the late universe (between

decoupling and the present day, as well as in the Hot Big Bang era itself). However inflation violates the timelike convergence condition at early times and so can avoid the initial singularity that would otherwise be predicted because these CTSs exist at later times. Theorems 2 and 3 fall away because of the timelike energy condition violation. If $k = 0$ or -1 , then a singularity will indeed occur, because of the Friedmann equation in the exact FL case [10], and in the perturbed FL case because of Theorem 1, relying only on the null energy condition, together with the existence of open space sections if they have their normal topology. Singularities are not inevitable when $k = +1$, both because the Friedmann equation now allows a minimum and because there are then closed spacelike sections so none of the above theorems apply. Various kinds of non-singular model can then occur [6]

4 The de Sitter Universe

We deal in turn with the three RW frames for de Sitter spacetime (see [11, 12, 3] for its global properties).

4.1 The $k=+1$ Frame

In the global $k = +1$ frame, the metric is (3) with $S(t) = A \cosh Ht$, $f(r) = \sin r$, where $A, H > 0$ and $-\infty < t < \infty$. The cosmology is non-singular and geodesically complete. Note that $0 \leq r \leq \pi$. The antipodal point to the origin of coordinates is at $r = \pi$; the equator for these coordinates is at $r = \pi/2$. Then by (4)

$$K_{;a}^a = \frac{2}{A^2 \cosh^2 Ht} \left[-AH \sinh Ht \pm \frac{\cos r}{\sin r} \right]$$

We get a past closed trapped surface if $t > 0$ and $AH \sinh Ht > |\frac{\cos r}{\sin r}|$. Now the latter term is zero at $r = \pi/2$ and diverges positively and negatively respectively at $r = 0, \pi$. Thus provided $t > 0$, we have a past closed trapped surface for all r such that

$$AH \sinh Ht > |\cot r|$$

which defines a non-zero set of 2-spheres around the equator at $r = \pi/2$. As $t \rightarrow 0$, these shrink to just the equator; as $t \rightarrow \infty$, they expand to a large part of the whole sphere. The origin is arbitrary, so every 2-sphere (t, r) constant with $t > 0$ and area greater than

$$A_* = 4\pi A^2 \cosh^2 Ht \sin^2(r_*), \quad AH \sinh Ht = \cot r_*$$

in any $k = +1$ frame will be a past closed trapped surface. Its normals will self-intersect and have caustics where $K_{;a}^a \rightarrow \infty$ at both $r \rightarrow 0$ (the origin, the ingoing family) and $r \rightarrow \pi$ (the antipodal point, the outgoing family). Since $\sin^2 r = \frac{1}{1+\cot^2 r}$ this gives

$$A_* = 4\pi \frac{A^2 \cosh^2 Ht}{1 + A^2 H^2 \sinh^2 Ht}.$$

As $t \rightarrow 0$, $S(t) \rightarrow A^2$ and $A_* \rightarrow 4\pi A^2$ (corresponding to $r \rightarrow \pi/2$); as $t \rightarrow \infty$, $S(t) \rightarrow \infty$ and $A_* \rightarrow 4\pi \frac{1}{H^2}$, so there is a minimum radius that will give a CTS (but the size of the 3-spaces increases without limit, so this minimum radius will be an ever smaller fraction of the size of the space sections). For $t < 0$ we find the corresponding family of future trapped surfaces. There are no trapped surface for $t = 0$.

For these trapped surfaces (2) becomes

$$\begin{aligned} \frac{d\theta}{dv} &= -\frac{1}{2}\theta^2 \Rightarrow \frac{d\theta}{\theta^2} = -\frac{1}{2}dv \Rightarrow \frac{1}{\theta} = -\frac{1}{2}(v - v_0) \\ &\Rightarrow \theta = -\frac{2}{v} \text{ on choosing } v_0 = 0. \end{aligned}$$

Thus the geodesics generating the pasts of the set of past trapped surfaces locally self-intersect, hence signalling an end to the boundary of the past of the trapped surfaces. But these intersections occur round the back (near the antipodal points); hence the past of these surfaces is not trapped by these null geodesics. Theorem 1 does not apply, even though the null energy condition is true, because the spatial sections are compact.

Note that these space sections are not unique [12]: every 3-sphere passing through the de Sitter throat is equivalent to every other one. The same result as above must be found in every such frame. This at first seems to lead to an apparent contradiction: no consistency is found at the same points on the hyperboloid in different frames, because in the $t = 0$ frames of different such choices there are no trapped surfaces but at the same points in the $t \neq 0$ frames there do exist closed trapped surfaces. However *the correspondence between events on the hyperboloid and 2-spheres depends on the frame chosen*. The frame corresponds point by point to events on the hyperboloid in each coordinate frame in a way such that the corresponding 2-spheres (spherically surrounding the origin of coordinates) depends on the coordinate frame chosen. *Given a specific choice of frame, however, a unique such correspondence of points and 2-spheres exists*. Those found in the $t = 0$ surfaces in a particular frame may be represented as marginally trapped 2-spheres in that frame, but are fully trapped in other frames. Thus one must choose a specific frame and work it all out in that frame; the results in all other frames will then follow by boosting, rotating and translating that frame.

4.2 Perturbed de Sitter Universes

The further basic point is that existence of CTSs do not imply singularities in perturbed de Sitter universes either, when the null generic condition holds, even when $\mu + p \geq 0$ (so that self-intersections occur because of the generic conditions as outlined above) but provided still $\mu + 3p < 0$. This is because they don't imply them in the de Sitter case, where the past of each CTS is also compact, but that does not imply a singularity because of the closed space sections. If this were not true the de Sitter universe would be unstable - but it is well known to be stable.

4.3 The $k=0$ frame

In the $k = 0$ frame, which covers half the spacetime (and so is not geodesically complete), we have the metric (3) with $S(t) = A \exp Ht$, $f(r) = r$, for $-\infty < t < \infty$ and $A, H > 0$. In this case (4) shows

$$K^a_{;a} = \frac{2}{A^2 \exp 2Ht} \left[-AH \exp Ht \pm \frac{1}{r} \right]$$

The second term dies away to zero, so there will be closed trapped surfaces for

$$AH \exp Ht > \frac{1}{r}$$

which will always be true for large enough r for any t . These will be part of the same set of 2-spheres as characterised above, but expressed in different coordinates. In this case they do correspond to geodesic incompleteness, because this coordinate frame does not cover the whole hyperboloid and Theorem 1 above applies in this frame. New information could come in from the other half of the hyperboloid if the solution is extended further (which even though it lies beyond the infinite redshift surface need not correspond to infinite redshift of matter beyond that surface; that would depend on how matter is moving in this further part of the hyperboloid). The spacetime is singular but extendible.

4.4 The $k=-1$ frame

In the $k = -1$ frame, which covers less than half the spacetime (and so is not geodesically complete), we have the metric (3) with $S(t) = A \sinh Ht$, $f(r) = \sinh r$, for $0 < t < \infty$ and $A, H > 0$. In this case (4) shows that

$$K^a_{;a} = \frac{2}{A^2 \sinh^2 Ht} \left[-AH \cosh Ht \pm \frac{\cosh r}{\sinh r} \right]$$

The latter term is always of magnitude > 1 , diverging as $r \rightarrow 0$ and $\rightarrow 1$ as $r \rightarrow \infty$. Thus there will be closed trapped surface for values of time t such that

$$AH \cosh Ht > 1, \quad t > 0 \Leftrightarrow t > (1/H) \arg \cosh(1/AH)$$

which will exist for all $A, H > 0$. For those values of t , closed trapped surfaces exist for all values of r such that

$$AH \cosh Ht > \coth r$$

which will then exist for large enough r . These will again be part of the same set of 2-spheres as characterised above, but expressed in different coordinates. Here again Theorem 1 predicts an initial singularity and again they are geodesically incomplete but extendible.

4.4.1 Conclusion: de Sitter spacetime

In all three cases, we show that *past closed trapped surfaces exist in the de Sitter universe* (of course these are just the same set of 2-surfaces found in different coordinates) and lead to *the boundaries of the pasts of those 2-surfaces being compact*. This does not however lead to spacetime singularities in the first case (de Sitter spacetime is geodesically complete in the $k = +1$ frame), because the pasts of the trapped 2-spheres are not trapped by these null boundaries, rather they can escape freely to earlier times because the 3-spaces are compact (the null cone intersections take place at the antipodal point on the other side of the 3-spaces with 3-sphere topology, allowing any interior matter to escape on this side, near the origin). However in the $k = 0$ and $k = -1$ frames the cosmology is singular because the worldlines do not cover the whole spacetime; indeed they self-intersect at $t = 0$ in the $k = -1$ case.

5 Emergent Universes

These are non-singular models with $k = +1$ that start off asymptotically as Einstein Static universes, and then evolve to de Sitter universes (and at even later times to a standard hot big bang) [6]. Here $k = +1$ and in the simplest case $S(t) = A + B \exp Ht$, with $-\infty < t < \infty$, and $A, B > 0$ [14]. Then by (4)

$$K_{;a}^a = \frac{2}{(A + B \exp Ht)^2} \left[-BH \exp Ht \pm \frac{\cos r}{\sin r} \right]$$

We get a past closed trapped surface if $BH \exp Ht > \left| \frac{\cos r}{\sin r} \right|$. Now the latter is zero at $r = \pi/2$ and diverges to $-\infty, \infty$ at $r = 0, \pi$ respectively. Thus provided $t > 0$, we have a closed trapped surface for all r such that

$$BH \exp Ht > |\cot r|$$

which defines a non-zero set of 2-spheres around the equator, depending on the coordinate time t . As $t \rightarrow -\infty$, the static limit, these shrink to just the equator; as $t \rightarrow \infty$, they expand to a large part of the whole sphere (as in the de Sitter case). However the spacetime is geodesically complete and non-singular. The past generators of the 2-spheres intersect, and this does not imply the existence of singularities; none of the singularity theorems apply. The de Sitter universe is the special case when $A = 0$, obtained as a completely smooth limit.

5.1 Perturbed Emergent universes

The existence of CTSs does not imply singularities in perturbed emergent universes either. Perturbations that ensure the genericity condition on all null geodesics again do not imply a singularity, as in the case of the de Sitter universe.

6 The Einstein Static Universe

Here we have a RW metric (3) with $S(t) = S_0 > 0$, $k = +1$, $-\infty < t < \infty$, and so

$$K_{;a}^a = \frac{2}{S^2(t)} \left[-\dot{S}(t) \pm \frac{\partial f / \partial r}{f(r)} \right] = \frac{2}{S_0^2} \left[0 \pm \frac{\cos r}{\sin r} \right].$$

In this case there are no closed trapped surfaces; however marginally trapped surfaces occur on the equator, the null geodesics generating their pasts intersecting at the antipodal point. Roughly: the closure of the space sections does not allow existence of 2-spheres that are large enough to be trapped. . This lack of closed trapped surfaces is connected to the high degree of stability of the E-S universe (they are stable to all inhomogeneous perturbations in the radiation case [13]). Again it is thus true that perturbing the universe leads to the generic condition for null geodesics but not to singularities.

7 Conclusion

Closed trapped surfaces occur in most Friedmann models, including the de Sitter universe. They necessarily lead to a singularity only if $\rho + 3p > 0$. When $\rho + 3p < 0$ and $k = +1$, singularity avoidance is possible. The null energy condition $\rho + p > 0$ does not necessarily lead to a singularity, despite existence of these closed trapped surfaces and hence compact boundaries of the past of these 2-spheres, when the spatial sections are compact.

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